Data Structures I
Advanced Functional Programming

Andres Löh (andres@cs.uu.nl)

Universiteit Utrecht

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Overview

Introduction (Lists)

Arrays

Unboxed types

Queues and deques

Summary and next lecture
Overview

Introduction (Lists)

Arrays

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Summary and next lecture
What is the most frequently used data structure in Haskell?

Clearly, lists ...
What is the most frequently used data structure in Haskell?

Clearly, lists . . .
What are lists good for?

- `head :: [a] → a`
- `tail :: [a] → [a]`
- `(_: :: a → [a] → [a]`

- These are efficient operations on lists.
- These are the stack operations.
What are lists good for?

- These are efficient operations on lists.
- These are the stack operations.

\[
\begin{align*}
    \text{head} &:: [a] \rightarrow a \\
    \text{tail} &:: [a] \rightarrow [a] \\
    (:) &:: a \rightarrow [a] \rightarrow [a]
\end{align*}
\]
What are lists good for?

\[
\begin{align*}
\text{head} &:: \ [a] \rightarrow a \quad \text{-- } O(1) \\
\text{tail} &:: \ [a] \rightarrow [a] \quad \text{-- } O(1) \\
(\cdot) &:: a \rightarrow [a] \rightarrow [a] \quad \text{-- } O(1)
\end{align*}
\]

- These are efficient operations on lists.
- These are the stack operations.
What are lists good for?

\[
top :: [a] \rightarrow a \quad -- \quad O(1)
\]

\[
\text{pop} :: [a] \rightarrow [a] \quad -- \quad O(1)
\]

\[
\text{push} :: a \rightarrow [a] \rightarrow [a] \quad -- \quad O(1)
\]

- These are efficient operations on lists.
- These are the stack operations.
Haskell stacks are persistent

A data structure is called **persistent** if after an operation both the original and the resulting version of the data structure are available.

If not persistent, a data structure is called **ephemeral**.

- Functional data structures are naturally persistent.
- Imperative data structures are usually ephemeral.
- Persistent data structures are often, but not always, less efficient than ephemeral data structures.
Haskell stacks are persistent

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Other operations on lists

\[ \text{snoc} :: [a] \rightarrow a \rightarrow [a] \quad \text{-- } O(n) \]
\[ \text{snoc} = \lambda xs \ x \rightarrow xs \mathbin{\mathbin{+\!\!\!\!\!\!\!\!\!\!\_}} [x] \]
\[ (\mathbin{\mathbin{+\!\!\!\!\!\!\!\!\!\!\_}}) :: [a] \rightarrow [a] \rightarrow [a] \quad \text{-- } O(n) \]
\[ \text{reverse} :: [a] \rightarrow [a] \quad \text{-- } O(n), \text{ naively: } O(n^2) \]
\[ \text{union} :: \text{Eq } a \Rightarrow [a] \rightarrow [a] \rightarrow [a] \quad \text{-- } O(mn) \]
\[ \text{elem} :: \text{Eq } a \Rightarrow a \rightarrow [a] \rightarrow \text{Bool} \quad \text{-- } O(n) \]

Although not efficient for these purposes, Haskell lists are frequently used as

- arrays
- queues, deques, catenable queues
- sets
- lookup tables, association lists, finite maps
- ...

Why?
Other operations on lists

\[ \text{snoc} :: [a] \rightarrow a \rightarrow [a] \quad -- O(n) \]
\[ \text{snoc} = \lambda xs \ x \rightarrow xs \uplus [x] \]
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Why?
Lists are everywhere, because …

- There is a convenient built-in notation for lists.
- There are even list comprehensions in Haskell.
- Lots of library functions on lists.
- Pattern matching!
- Haskell strings are lists.
- Other data structures not widely known.
- Arrays are often worse.
- Not enough standard libraries for data structures.

We are going to change this …

Unfortunately, the remaining reasons are valid.
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Unfortunately, the remaining reasons are valid.
Therefore, a bit of advice

Use accumulating arguments.

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\text{reverse} & \quad :: \ [a] \rightarrow [a] \\
\text{reverse} & \quad = \ \text{reverse}' \ [ ] \\
\text{reverse}' & \quad :: \ [a] \rightarrow [a] \rightarrow [a] \\
\text{reverse}' \ \text{acc} \ [ ] & \quad = \ \text{acc} \\
\text{reverse}' \ \text{acc} \ (x : xs) & \quad = \ \text{reverse}' \ (x : \text{acc}) \ xs
\end{align*}
\]

The concatenation trick.

Compare:

\[
(((xs_1 + +) \cdot (xs_2 + +)) \cdot (xs_3 + +)) \cdot (xs_4 + +)) \quad "" \\
(((xs_1 + xs_2) + xs_3) + xs_4)
\]
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\text{reverse}' \text{ acc} (x : xs) & = \text{reverse}' (x : \text{acc}) \text{ xs}
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About arrays

Imperative arrays feature

▶ constant-time lookup
▶ constant-time update

Update is usually at least as important as lookup.

Functional arrays do

▶ lookup in $O(1)$; yay!
▶ update in $O(n)$! Why? Persistence!

Array update is even worse than list update.

▶ To update the $n$th element of a list, $n - 1$ elements are copied.
▶ To update any element of an array, the whole array is copied.
About arrays

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Space efficiency vs. space leaks

Arrays can be stored in a compact way. Lists require lots of pointers.

If arrays are updated frequently and used persistently, space leaks will occur!
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Mutable arrays

- Are like imperative arrays.
- Defined in `Data.Array.MArray` and `Data.Array.IO`.
- All operations in a state monad (possibly IO monad).
- Often awkward to use in a functional setting.
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Haskell data structures are **boxed**.

- Each value is behind an additional indirection.
- This allows polymorphic data structures (because the size of a pointer is always the same).
- This allows laziness, because the pointer can be to a computation as well as to evaluated data.

GHC offers **unboxed** datatypes, too. Naturally, they

- are slightly more efficient (in both space and time),
- are strict,
- cannot be used in polymorphic data structures.
Boxed vs. unboxed types

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▶ are slightly more efficient (in both space and time),
▶ are strict,
▶ cannot be used in polymorphic data structures.
Unboxed types

- Defined in GHC.Base.
- For example, Int#, Char#, Double#.
- Have kind #, not ∗.
- Use specialized operations such as

  \((+\#) :: \text{Int}\# \rightarrow \text{Int}\# \rightarrow \text{Int}\#\)

- Cannot be used in polymorphic functions or datatypes.
- Are used by GHC internally to define the usual datatypes:

  ```haskell
  data Int = I# Int#
  ```
Packed strings

- Defined in `Data.PackedString`.
- Implemented as immutable, unboxed arrays.
- Can be more space-efficient than standard strings.
- Manipulating packed strings can be expensive.
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Queues

- Stacks are LIFO (last-in-first-out).
- Queues are FIFO (first-in-first-out).
- A list is not very suitable to represent a queue, because efficient access to both ends is desired.

The standard trick is:

```plaintext
data Queue a = Q [a] [a]
```

The first list is the front, the second the back of the queue, in reversed order.
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The standard trick is:

```
data Queue a = Q [a] [a]
```

The first list is the **front**, the second the **back** of the queue, in reversed order.
Queue operations

This is what we want for a queue:

- **empty**: `:: Queue a` -- produce an empty queue
- **snoc**: `:: a → Queue a → Queue a` -- insert at the back
- **head**: `:: Queue a → a` -- get first element
- **tail**: `:: Queue a → Queue a` -- remove first element
- **toList**: `:: Queue a → [a]` -- queue to list
- **fromList**: `:: [a] → Queue a` -- list to queue
Queue operations

This is what we want for a queue:

- **empty** :: Queue $a$  
  -- produce an empty queue
- **snoc** :: $a \rightarrow$ Queue $a \rightarrow$ Queue $a$  
  -- insert at the back
- **head** :: Queue $a \rightarrow a$  
  -- get first element
- **tail** :: Queue $a \rightarrow$ Queue $a$  
  -- remove first element

- **toList** :: Queue $a \rightarrow [a]$  
  -- queue to list
- **fromList** :: $[a] \rightarrow$ Queue $a$  
  -- list to queue
Implementing queue operations

\[\textit{empty} :: \text{Queue}\ a \]
\[\textit{empty} = Q \ [\ ] \ [\ ]\]

\[\textit{snoc} :: a \rightarrow \text{Queue}\ a \rightarrow \text{Queue}\ a\]
\[\textit{snoc} \ x \ (Q \ fs \ bs) = Q \ fs \ (x : bs)\]
Implementing queue operations

\[ \text{empty :: Queue } a \]
\[ \text{empty} = Q \text{ [] []} \]

\[ \text{snoc :: } a \rightarrow \text{Queue } a \rightarrow \text{Queue } a \]
\[ \text{snoc } x \ (Q \text{ fs bs}) = Q \text{ fs } (x : bs) \]
Invocations of \textit{reverse}

\begin{align*}
\textit{head} :: \text{Queue} \ a & \to a \\
\textit{head} \ (Q \ (f : fs) \ bs) & = f \\
\textit{head} \ (Q \ [] \ bs) & = \textit{head} \ (\text{reverse} \ bs)
\end{align*}

\begin{align*}
\textit{tail} :: \text{Queue} \ a & \to \text{Queue} \ a \\
\textit{tail} \ (Q \ (f : fs) \ bs) & = Q \ fs \ bs \\
\textit{tail} \ (Q \ [] \ bs) & = Q \ (\text{tail} \ (\text{reverse} \ bs)) \ []
\end{align*}

Persistence spoils the fun here; without persistence, all operations would be in \textit{O}(1) amortized time.
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\text{tail} & (Q (f : fs) bs) = Q \ fs \ bs \\
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\end{align*}

Persistence spoils the fun here; without persistence, all operations would be in \(O(1)\) amortized time.
Amortized analysis

Amortized complexity can be better than worst-case complexity if the worst-case cannot occur that often in practice.

In an amortized analysis, we look at the cost of multiple operations rather than single operations.
Idea

- Distribute the work that *reverse* causes over multiple operations in such a way that the amortized cost of each operation is constant.
- Use laziness (and memoization) to ensure that expensive operations are not performed too early or too often.
Memoization

A suspended expression in a lazy language is evaluated only once. The suspension is then updated with the result. Whenever the same expression is needed again, the result can be used immediately. This is called memoization.
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Efficient queues

Recall the queue representation:

```
data Queue a = Q [a] [a]
```

As we will see, the work of reversing the list can be distributed well by choosing the following invariant for $Q \ fs \ bs$:

```
length fs >= length bs
```

In particular, $length \ fs = 0$ if and only if the queue is empty.

We need the lengths of both lists available in constant time.
Efficient queues

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Efficient queues

Recall the queue representation:

```
data Queue a = Q ! Int [a] ! Int [a]
```

As we will see, the work of reversing the list can be distributed well by choosing the following invariant for \( Q \lf \; fs \; lb \; bs \):

```
lf \geq lb
```

In particular, \( \text{length} \; fs = 0 \) if and only if the queue is empty.

We need the lengths of both lists available in constant time.
empty and head are simple due to the invariant

\[
\text{empty} :: \text{Queue } a \\
\text{empty} = Q \ 0 \ [] \ 0 \ []
\]

\[
\text{head} :: \text{Queue } a \rightarrow a \\
\text{head} \ (Q \ (f : fs) \ bs) = f \\
\text{head} \ (Q \ [] \ _) = \text{error "empty queue"}
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\text{empty :: Queue } a \\
\text{empty} = \text{Queue } 0 [] 0 []
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\text{head :: Queue } a \rightarrow a \\
\text{head } (\text{Queue } f : fs \ b s) = f \\
\text{head } (\text{Queue } [] \text{ _}) = \text{error } "\text{empty queue}" 
\]
What about *tail* and *snoc*?

\[
\begin{align*}
tail :: \text{Queue } a & \rightarrow a \\
tail (Q \text{ lf } (f : fs) \text{ lb } b) &= \text{makeQ } (\text{lf } - 1) \text{ fs } lb \text{ b} \\
tail (Q - [ ] - _) &= \text{error } "\text{empty queue}" \\
\end{align*}
\]

\[
\begin{align*}
\text{snoc} :: a & \rightarrow \text{Queue } a \rightarrow \text{Queue } a \\
\text{snoc } x (Q \text{ lf } fs \text{ lb } bs) &= \text{makeQ } \text{lf } fs (\text{lb } + 1) (x : bs)
\end{align*}
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In both cases, we have to make a new queue using a call \(\text{makeQ } \text{lf } f \text{ lb } f'\), where we may need to re-establish the invariant.
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In both cases, we have to make a new queue using a call \( \text{makeQ} \ lf \ f \ lb \ f' \), where we may need to re-establish the invariant.
How to make a queue

\[
\text{makeQ} :: \text{Int} \rightarrow [a] \rightarrow \text{Int} \rightarrow [a] \\
\rightarrow \text{Queue } a \\
\text{makeQ } lf \quad fs \quad lb \quad bs \\
| \text{lf } \geq \text{lb} \quad = \quad \text{Q } lf \quad fs \quad lb \quad bs \\
| \text{otherwise } = \quad \text{Q } (lf + lb) \quad (fs ++ \text{reverse } bs) \quad 0 \quad []
\]
Why is this implementation "better"?

(drawing and lots of handwaving)

Read Okasaki’s book for a proof.
Why is this implementation “better”?  

(drawing and lots of handwaving)  

Read Okasaki’s book for a proof.
Queues in GHC

- Available in Data.Queue (ghc-6.4).
- Based on a slight variation of the implementation described here, allowing operations in constant worse-case time ("real-time queues").
- Representation of queues is then

```haskell
data Queue a = Q [a] [a] [a]
```

where the third list is used to maintain the unevaluated part of the front queue.

- Also described in Okasaki’s book.
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Deques

A deque is a double-ended queue.

Operations like queue operations:

- `empty :: Deque a` -- produce an empty queue
- `snoc :: a → Deque a → Deque a` -- insert at the back
- `head :: Deque a → a` -- get first element
- `tail :: Deque a → Deque a` -- remove first element
- `toList :: Deque a → [a]` -- queue to list
- `fromList :: [a] → Deque a` -- list to queue

Additionally (also in constant time):

- `cons :: a → Deque a → Deque a` -- insert at the front
- `init :: Deque a → Deque a` -- remove last element
- `last :: Deque a → a` -- get last element
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Additionally (also in constant time):

- **cons** :: a → Deque a → Deque a -- insert at the front
- **init** :: Deque a → Deque a -- remove last element
- **last** :: Deque a → a -- get last element
The previous implementation can easily be extended to work for deques:

```
data Deque a = D ! Int [a] ! Int [a]
```

Of course, we have to make the representation more symmetric. The invariant for $D \ l f \ f s \ l b \ b s$ becomes:

```
lf \leq c*lb + 1 \land lr \leq c*lf + 1
```

(for some constant $c > 1$).
Implementation of deques

- The implementation of `makeQ` must be adapted to maintain this invariant.
- The other operations are straight-forward, we only have to pay attention to the one-element queue.
- How much time does it cost to reverse a deque?
- Unfortunately, there currently is no standard Haskell library for deques.
Catenable queues or deques

- Queues or deques that support efficient concatenation are called **catenable**.

- It is possible to support concatenation in $O(\log n)$ and even in $O(1)$ amortized time, but this requires a completely different implementation of queues/deques.

- Again, there currently are no standard Haskell libraries for catenable queues and deques.
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Summary and next lecture
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- Lists are everywhere in Haskell, for a lot of good reasons.
- Functional data structures are persistent.
- Persistence and efficiency and evaluation order all interact.
- Array updates are inherently inefficient in a functional language.
- Queues and deques support many operations efficiently that normal lists do not.
- In a persistent setting, queue and deque operations can be implemented with the same complexity bounds as in an ephemeral setting.
- GHC has a standard library that supports many (but not all desirable) datastructures, for instance lists, queues, arrays in all flavors, but also unboxed types and packed strings.
Next lecture

- Pattern matching, abstract datatypes, views.
- Trees, finite maps and sets.
- ...